Grade 6 Math Circles
March 7/8/9, 2023
Transformations

What is a Transformation?

A **transformation** describes the change in position or appearance of an object. Did you look at yourself in the mirror today? Did you get to and from school? Did you look at the hands of a clock moving? All of these are considered transformations. Transformations are everywhere and everyone witnesses and performs them, whether we're aware of it or not.

We can define the **object** as the original thing we are transforming. The **image** is what we call the object after it has been transformed. In this lesson, shapes shaded in dark gray will be the object and shapes outlined in black will be the image.

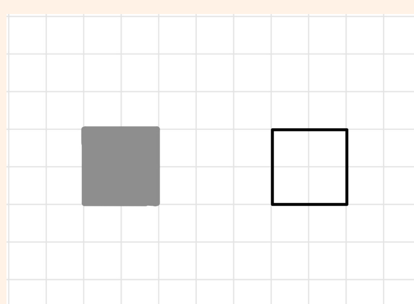
In this lesson, we will discover four different transformations and how they relate to math.

Translations

A **translation** describes the movement or shift of an object. Walking from one side of the room to another is an example of a translation. Another example would be sliding your Math Circles handout across the table. There are countless examples of translations we can think of but for now we will consider translations of shapes.

Example A

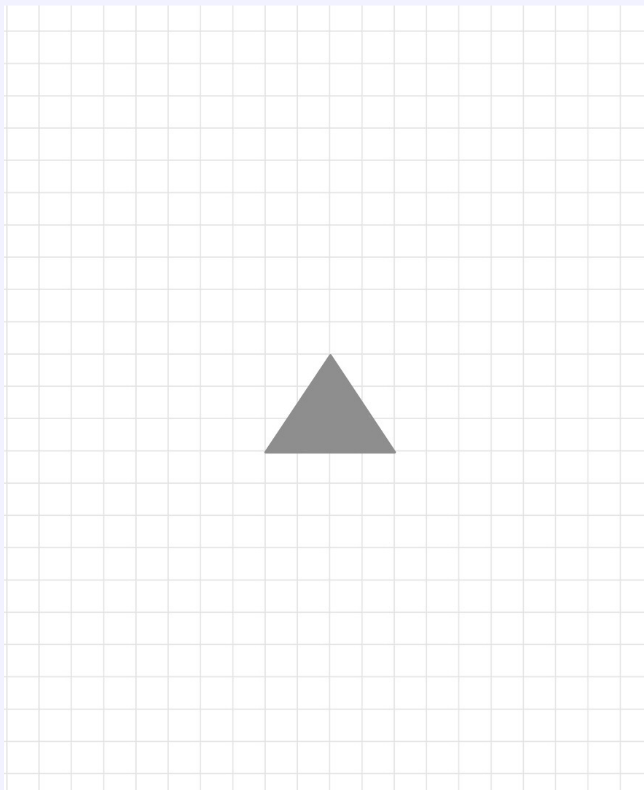
Translate the given square 5 units to the right.



Notice how we took each point of the object and shifted it 5 units to the right.

Exercise 1

Perform the following individual transformations on the triangle below. Each transformation should be applied to the original triangle, not an image from the previous part.



a) Translate 4 units up

b) Translate 3 units left

c) Translate 5 units down

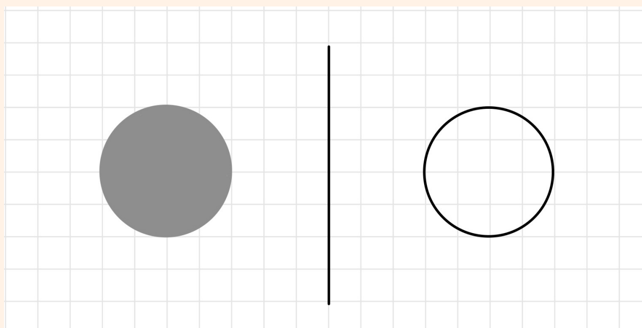
d) Translate 2 units right

Reflections

When we **reflect** an object, we are flipping it. You're probably quite familiar with the word reflection. The reflection of yourself you see in a mirror is a perfect example of the reflection transformation.

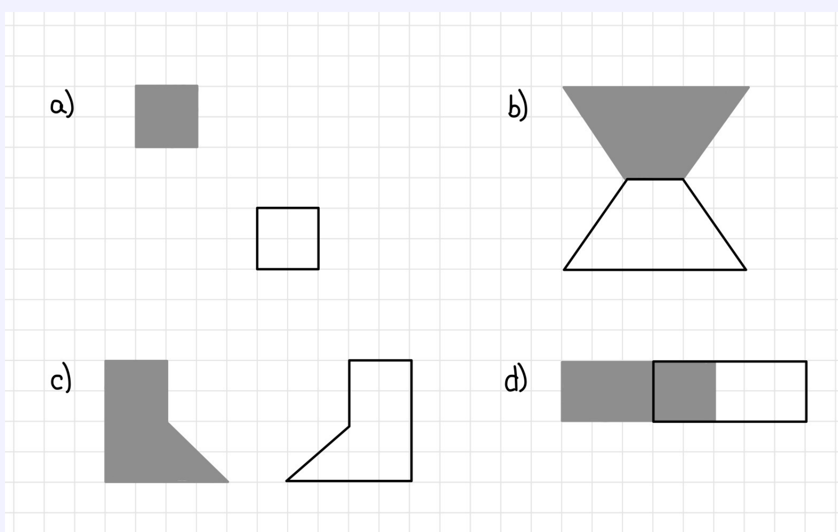
The image you see of yourself in the mirror is yourself flipped. When reflecting objects, we need something to reflect over called a **mirror line** (also called the **axis of reflection**). In the case of looking at your reflection in the mirror, the mirror line is the mirror itself.

Example B
Reflect the following circle over the given mirror line.



Exercise 2

For each of the following reflections, draw where the mirror line should go.



Rotations

A **rotation** of an object occurs when it is moved around a certain fixed point. We call this point the **centre of rotation**. Imagine your friends are standing in a circle and you are standing in the middle of the circle. If your friends all start walking around the circle, then your friends are rotating around you and you would be considered the centre of rotation. For rotations, we use the words **clockwise** (abbreviated to CW) and **counterclockwise** (abbreviated to CCW) to indicate which direction we are rotating. We also use **degrees** represented by ° to indicate how much we are rotating by.

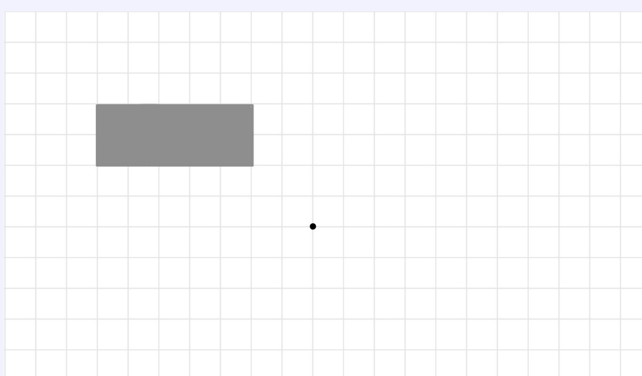
Stop and Think

How many degrees are there in one full rotation?

Let's take a look at this demo to better understand points of rotation.

Exercise 3

Rotate the following rectangle 180° CW around the given centre of rotation.

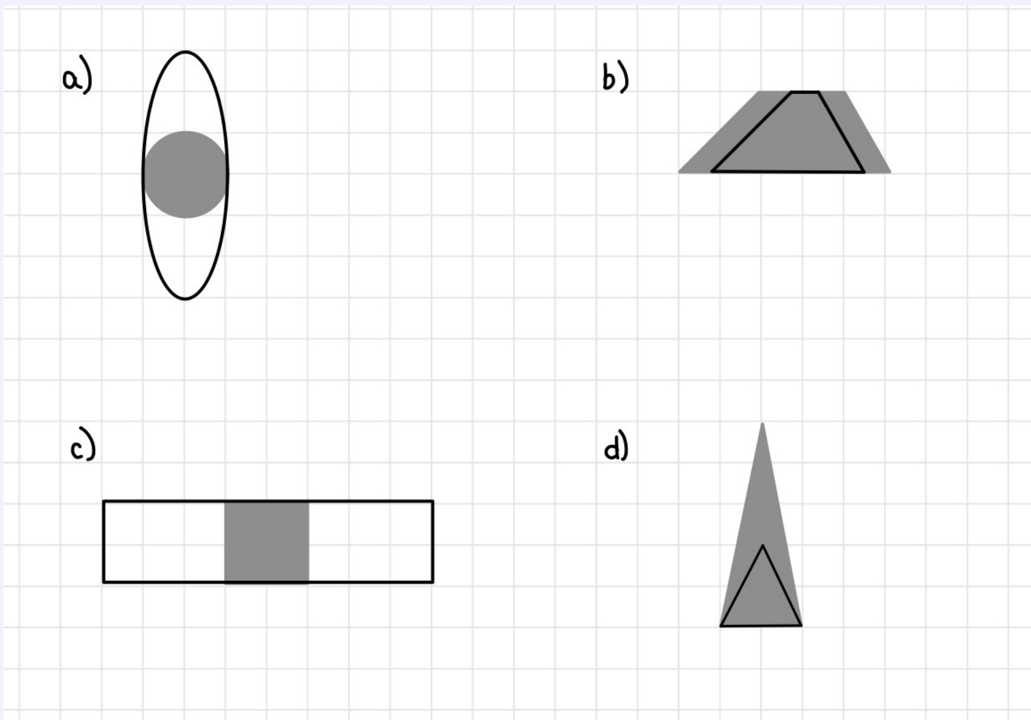


Stretches and Compressions

A **stretch** makes an object wider (or taller) and a **compression** makes it narrower (or shorter). If you've ever played with slime before, you've probably stretched it out or compressed it into a ball. There are two different types of stretches and compressions. The first is horizontal which we can think of as making an object fatter or skinnier. The second is vertical which we can think of as making an object taller or shorter.

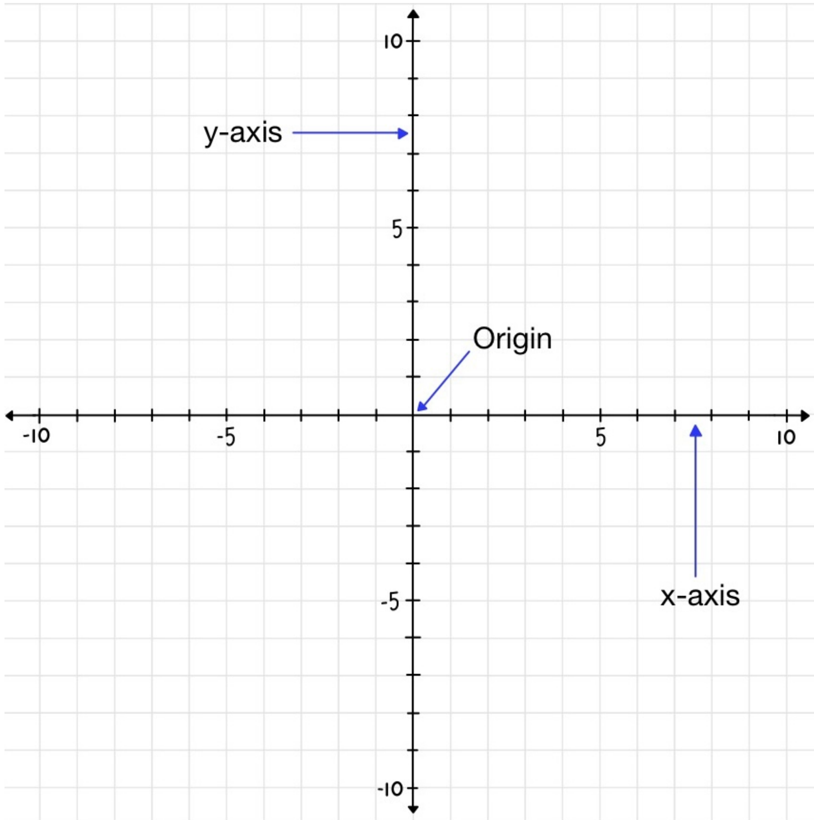
Exercise 4

Determine whether each object has been stretched or compressed. In addition, state whether it was a vertical or horizontal stretch/compression.



Cartesian Plane Review (from Week 1)

The Cartesian Plane is made up of two number lines: one running horizontally and the other running vertically. We call the horizontal number line the x**-axis** and the vertical number line is called the y**-axis**. The x-axis and the y-axis intersect where both of them are 0. We call this point of intersection the **origin**.



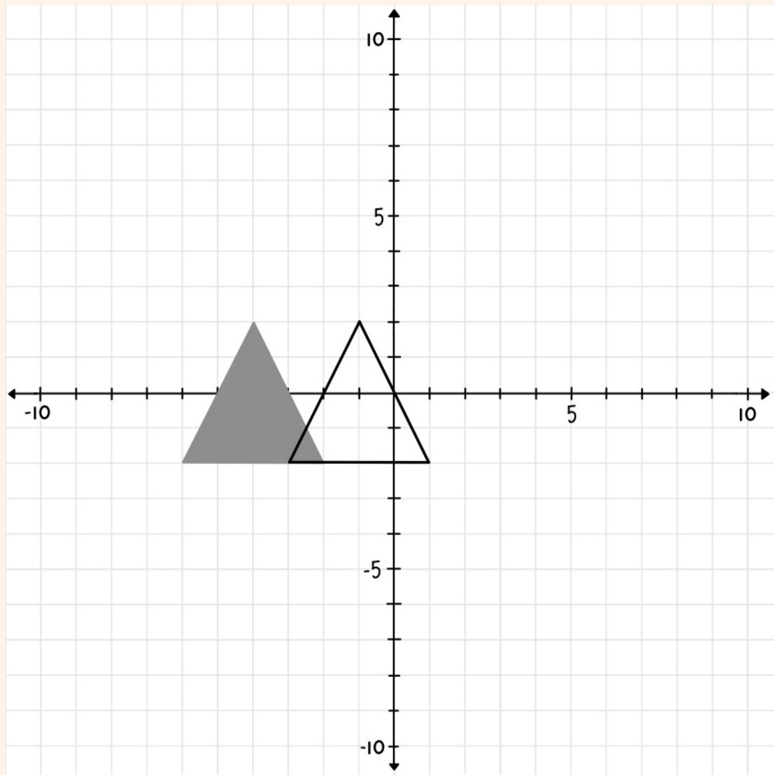
We can place points anywhere on the Cartesian Plane. Each point will have two numbers corresponding to it. The first one is called the x**-coordinate** and that number decides where along the x-axis that point will lay. Similarly, the second number is called the y**-coordinate** and it determines where along the y-axis the point will be. We write the coordinates of a point in brackets in the form (x-coordinate, y-coordinate).

Applications to Graphing

For the first part of the lesson, we worked with transforming shapes. In this next section, we are going to keep working with shapes but on a Cartesian plane and discover what happens to their coordinates when we perform certain transformations.

Example C

Take a look at the following object and its image. Determine which transformation was applied here. Write out some coordinates for the object and the image.



Object coordinates

x	-6	-5	-4	-3	-2	-4
y	-2	0	2	0	-2	-2

Image coordinates

x	-3	-2	-1	0	1	-1
y	-2	0	2	0	-2	-2

Stop and Think

Do you notice a pattern between the coordinates of the object and the image in the above example?

Since this was a horizontal translation (shifts to the left or right), we noticed that the x-coordinates changed but the y-coordinates stayed the same. In fact, the shift of 3 units to the right in the example above caused the x-coordinates to increase by 3.

Stop and Think

What would happen to the x-coordinate if we were to have a translation of 3 units to the left?

Exercise 5

Draw any shape you would like on the Cartesian plane. Draw an image of that shape with the translation of 2 units down. Write down a few coordinates of the object and the image.

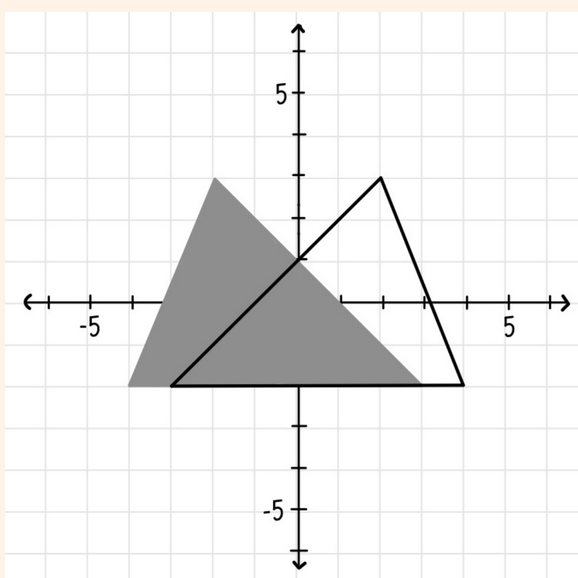
You probably noticed in the exercise that this time, the x-coordinate stayed the same and the y-coordinate was the one to change. In fact, the shift 2 down caused the y-coordinates to decrease by 2. Similarly, if we were to shift the object up 2 units, the y-coordinate would increase by 2.

Reflections

In the Cartesian plane, you can reflect objects over any line. However, in this lesson we are only going to be exploring reflections over the x-axis and the y-axis.

Example D

Take a look at the following object and its image reflected over the y-axis. Write down some coordinates for the object and the image.



Object coordinates

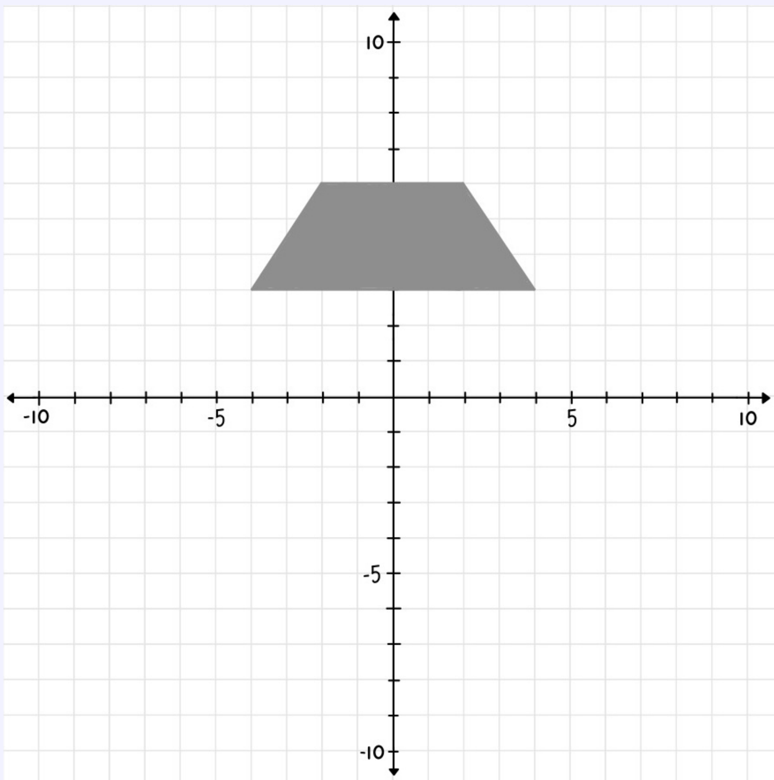
x	-4	-2	1	3	0
y	-2	3	0	-2	-2

Image coordinates

x	4	2	-1	-3	0
y	-2	3	0	-2	-2

Exercise 6

Reflect the following object over the x-axis and write some coordinates for the object and the image.



We conclude that a reflection over the x-axis results in the y-coordinate being negated (which means positive becomes negative and negative becomes positive) and a reflection over the y-axis results in the x-coordinate being negated.

Rotation

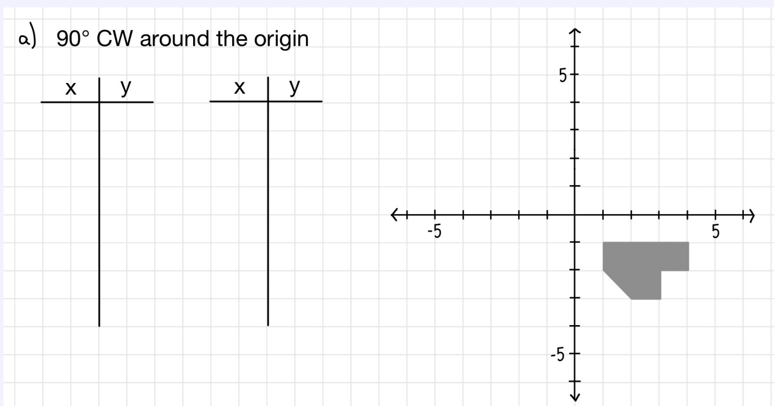
Rotations in the Cartesian plane can be done around any centre of rotation. For this lesson, we will only look at rotations about the origin. The effect that rotations have on the coordinates of points are not as obvious as other transformations so I have included a table below that describes what happens to the points during certain rotations.

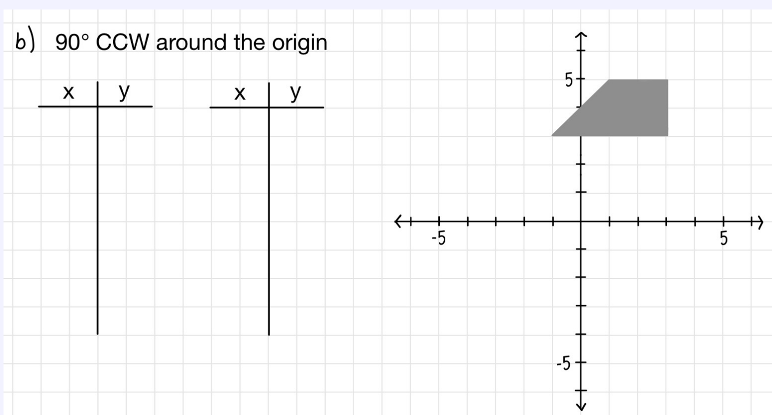
90° CW (or 270° CCW)	(x, y) becomes $(y, -x)$
180° (CW or CCW)	(x, y) becomes $(-x, -y)$
90° CCW (or 270° CW)	(x, y) becomes $(-y, x)$

There are obviously many more possible rotations than the ones shown above but for this lesson we will only cover these three.

Exercise 7

Rotate the following objects. Start by writing the coordinates of the object and using the table above to find the coordinates of the image. Then, sketch the image.



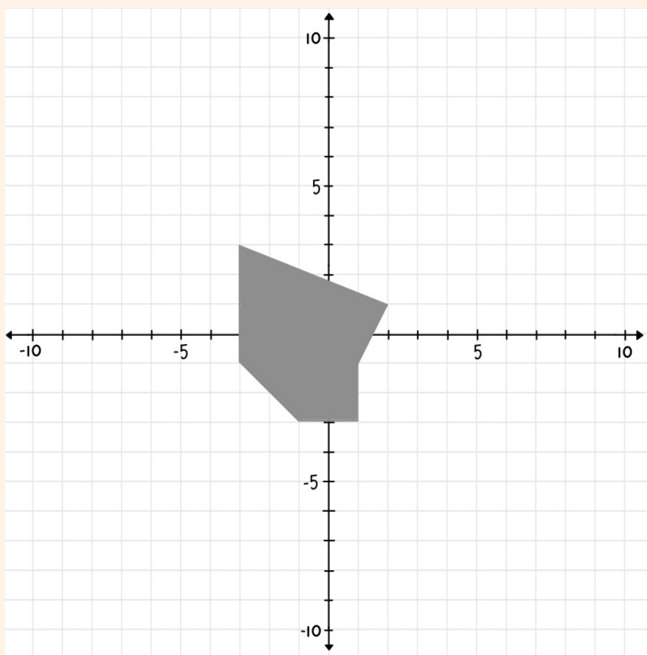


Stretches and Compressions

We measure stretches and compressions with factors which are numbers. When we say "stretch the object vertically by a factor of 2", what we mean is that the object becomes two times taller. When working on the Cartesian plane, it may be a bit difficult to determine the factor of the stretch or compression just by looking at the object and image. For this section, we are going to first perform transformations on points and then see what happens to the image.

Example E

Look at the following shape and its given coordinates. Multiply each x-coordinate by 3 and sketch the resulting coordinates.



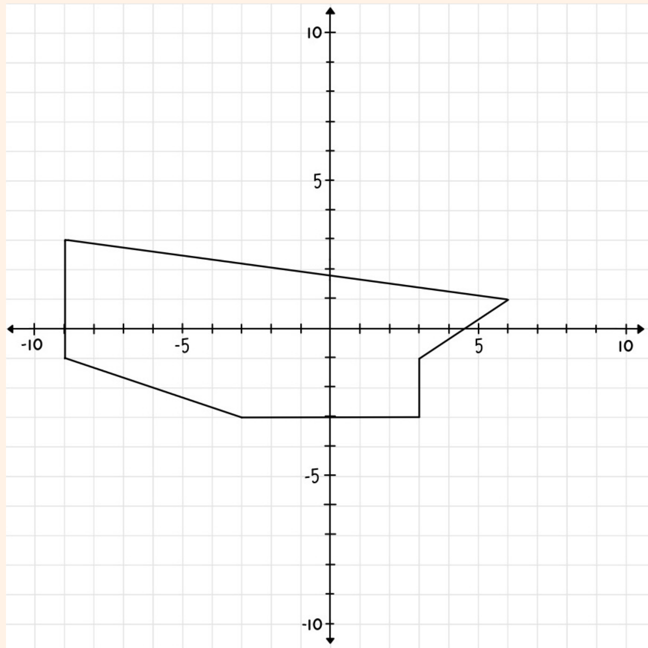
Object coordinates

x	-3	2	1	1	-1	-3
y	3	1	-1	-3	-3	-1

Image coordinates

x	-9	6	3	3	-3	-9
y	3	1	-1	-3	-3	-1

So the image will look like:



Stop and Think

What do you think would happen if we were to divide the x-coordinates by 3 instead of multiply?

Exercise 8

On the Cartesian plane, draw a shape of your choosing. Write out the coordinates and divide each y-coordinate by 2. Sketch the resulting image and observe what happened to the original object.

As we could see, multiplying the x-coordinate by a number results in a horizontal stretch by that number. We can also deduce that dividing the x-coordinate by a number results in a horizontal compression by that number. By this reasoning, we can also say the same thing for vertical stretches and compression in terms of the y-coordinate.

Putting it all Together

After learning about all these transformations, we will now look at how we can put them all together.

Stop and Think

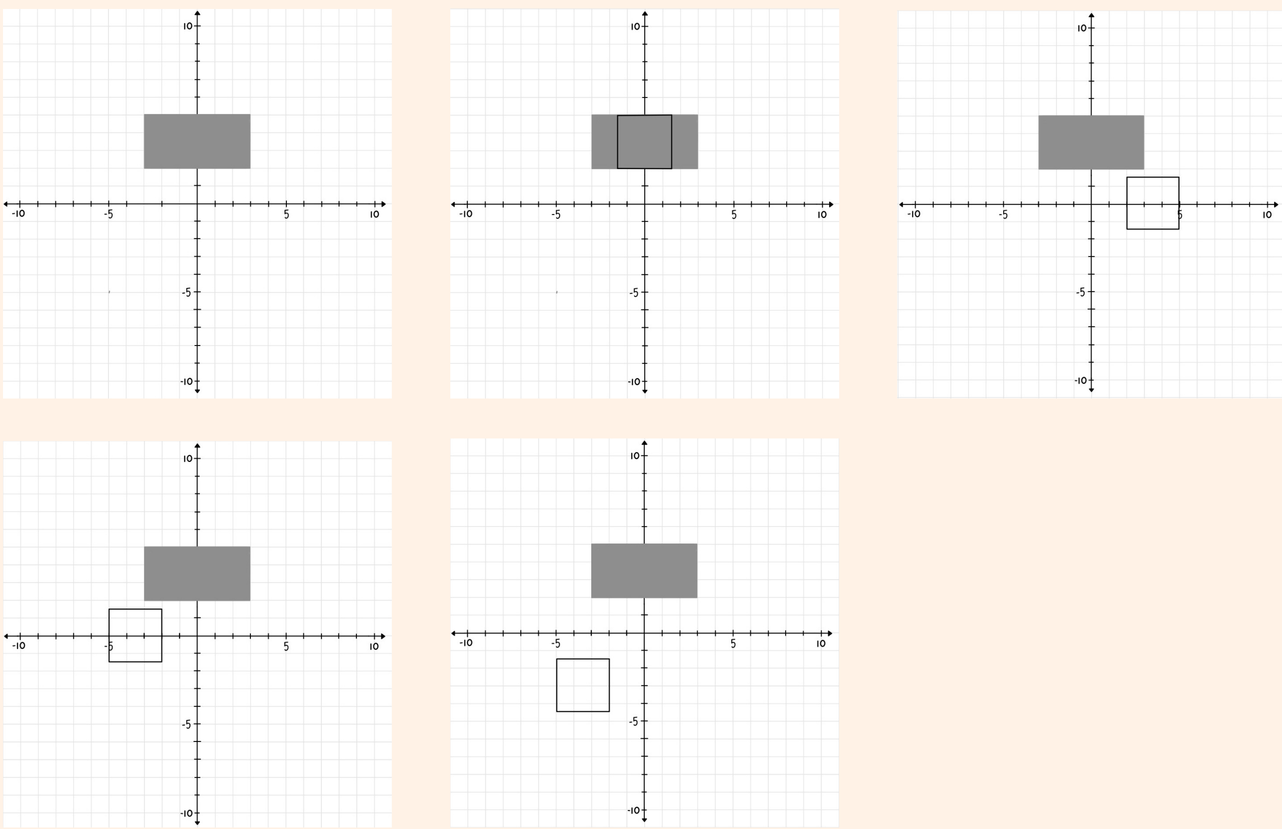
Do you think the order of performing transformations matters?

Let's look at an example to find out.

Example F

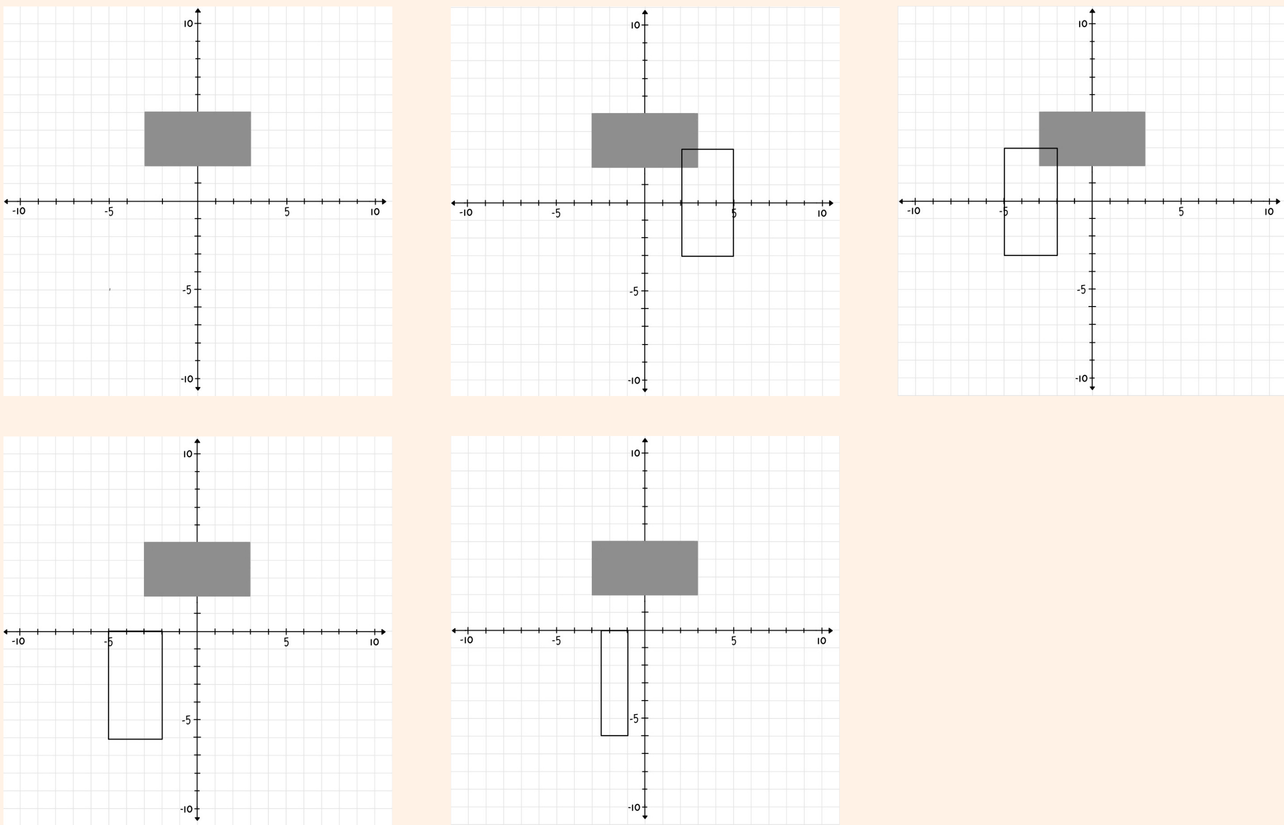
Given the rectangle below, we will perform the following transformations in order.

1. Horizontal compression by a factor of 2
2. Rotation 270° CCW about the origin
3. Reflection over y-axis
4. Shift 3 units down



Next, we will perform the same transformations but switch up the order.

1. Rotation 270° CCW about the origin
2. Reflection over y-axis
3. Shift 3 units down
4. Horizontal compression by a factor of 2

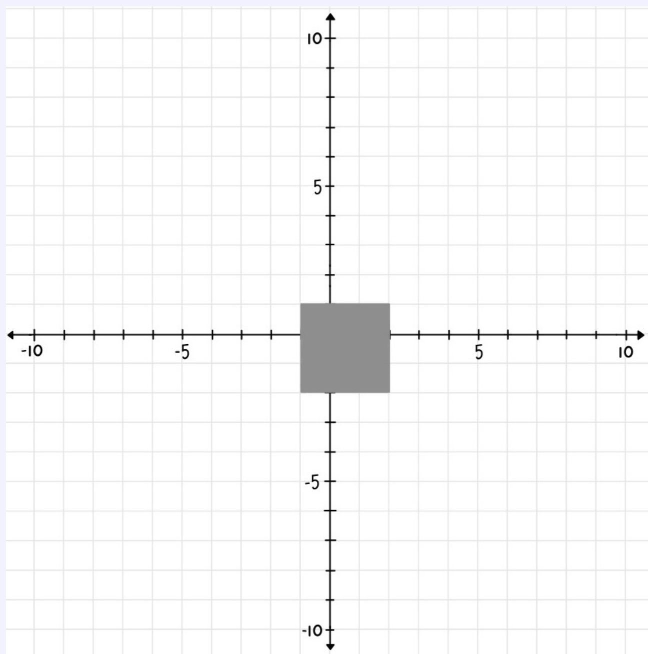


As we saw above, the order we perform transformation <u>does</u> matter. So it's important to remember to perform the transformations in the order that they're given, otherwise you could get a different image.

Exercise 9

Perform the following transformations in order on the shape below.

1. Reflection over y-axis
2. Shift 3 units up
3. Shift 4 units right
4. Rotation 90° CW about the origin
5. Vertical stretch by a factor of 2



A Summary of Transformations

Translation n units right	(x, y) becomes $(x + n, y)$
Translation n units left	(x, y) becomes $(x - n, y)$
Translation n units up	(x, y) becomes $(x, y + n)$
Translation n units down	(x, y) becomes $(x, y - n)$
Reflection over the x-axis	(x, y) becomes $(x, -y)$
Reflection over the y-axis	(x, y) becomes $(-x, y)$
Rotation 90° CW (or 270° CCW)	(x, y) becomes $(y, -x)$
Rotation 180° (CW or CCW)	(x, y) becomes $(-x, -y)$
Rotation 90° CCW (or 270° CW)	(x, y) becomes $(-y, x)$
Horizontal stretch by a factor of n	(x, y) becomes $(n \times x, y)$
Horizontal compression by a factor of n	(x, y) becomes $(\frac{x}{n}, y)$
Vertical stretch by a factor of n	(x, y) becomes $(x, n \times y)$
Vertical compression by a factor of n	(x, y) becomes $(x, \frac{y}{n})$